Raabe's Test

— let $\sum u_n$ be a positive term series and let $\lim n\left[\frac{u_n}{u_{n+1}} - 1\right] = l$

then the series is
i) Convergent if $l > 1$
2) divergent if $l < 1$
3) No firm decision is possible if $l = 1$

Proof:-

Case I:

When $l > 1$

let α be any number such that $1 < \alpha < l$

then $\exists$ m such that $\forall\, n \geq m$

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] > \alpha$$

$$\Rightarrow \quad n u_n - n u_{n+1} > \alpha u_{n+1}$$

$$\Rightarrow \quad n u_n - (n+1) u_{n+1} > (\alpha - 1) u_{n+1}$$

Replacing n by m, m+1, m+2 ...... n and adding, we have

$$\left[\begin{array}{l} m u_m - \cancel{(m+1) u_{m+1}} > (\alpha-1) u_{m+1} \\ \cancel{(m+1) u_{m+1}} - \cancel{(m+2) u_{m+2}} > (\alpha-1) u_{m+2} \\ \cancel{(m+2) u_{m+2}} - \cancel{(m+3) u_{m+3}} > (\alpha-1) u_{m+3} \\ \cancel{-}\ \cancel{-} - - - - - - - - - - \\ - - - - \cancel{-} - - - - - - - - - \\ \cancel{n u_n} - (n+1) u_{n+1} > (\alpha-1) u_{n+1} \end{array}\right]$$

$$\therefore \quad m u_m - (n+1) u_{n+1} > (\alpha-1) \left[u_{m+1} + u_{m+2} + \cdots + u_{n+1} \right]$$

i.e. $(\alpha-1) \left[u_{m+1} + u_{m+2} + \cdots + u_{n+1} \right] < m u_m$.

$$\Rightarrow u_{m+1} + u_{m+2} + \cdots + u_{n+1} < \frac{m u_m}{\alpha-1}$$

We have,

$$S_n = u_1 + u_2 + u_3 + \cdots + u_m + u_{m+1} + u_{m+2} + \cdots + u_n$$

$$S_n \leq \left[u_1 + u_2 + u_3 + \cdots + u_m \right] + \frac{m u_m}{\alpha-1} \quad \forall\ n \in \mathbb{N}.$$

So, the sequence $\langle S_n \rangle$ is bounded and as such the series is convergent

Case II:
when $l<1$

let α be any number such that
$l<\alpha<1$
then $\exists$ m such that $\forall$ $n \geq m$

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] < \alpha$$

It follows that $\forall$ $n \geq m$

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] < 1$$

$$\Rightarrow \frac{u_n}{u_{n+1}} < 1 + \frac{1}{n}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} < \frac{n+1}{n}$$

$$\Rightarrow \frac{u_{n+1}}{u_n} > \frac{n}{n+1}$$

Replacing n by m, $m+1$, $m+2$, ... $n-1$ and multiplying them, we get

$$\left[\begin{aligned} \frac{u_{m+1}}{u_m} &> \frac{m}{m+1} \\ \frac{u_{m+2}}{u_{m+1}} &> \frac{m+1}{m+2} \\ \frac{u_{m+3}}{u_{m+2}} &> \frac{m+2}{m+3} \\ &\cdots\cdots \\ &\cdots\cdots \\ \frac{u_n}{u_{n-1}} &> \frac{n-1}{n} \end{aligned}\right]$$

$$\Rightarrow \frac{u_n}{u_m} > \frac{m}{n} \quad \forall \ n \geq m$$

$$\Rightarrow u_n > \frac{K}{n} \quad \text{where, } K = m u_m.$$

Also, the series $\sum \frac{1}{n}$ is divergent

So, ~~$\nexists$~~ by Comparision Test of second Type, $\sum u_n$ is divergent.

Case III :
when $l=1$

Consider the two series

i) $\sum \frac{1}{n(\log n)^2}$ $(n \geq 2)$

2) $\sum \frac{1}{n}$

For both the series
$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = 1$$

But the first series is convergent while the second is divergent.